

# On Piercing Numbers of Families Satisfying the $(p, q)_r$ Property

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## Abstract

The Hadwiger-Debrunner number  $\text{HD}_d(p, q)$  is the minimal size of a piercing set that can always be guaranteed for a family of compact convex sets in  $\mathbb{R}^d$  that satisfies the  $(p, q)$  property. Hadwiger and Debrunner showed that  $\text{HD}_d(p, q) \geq p - q + 1$  for all  $q$ , and equality is attained for  $q > \frac{d-1}{d}p + 1$ . Almost tight upper bounds for  $\text{HD}_d(p, q)$  for a ‘sufficiently large’  $q$  were obtained recently using an enhancement of the celebrated Alon-Kleitman theorem, but no sharp upper bounds for a general  $q$  are known.

In [9], Montejano and Soberón defined a refinement of the  $(p, q)$  property:  $\mathcal{F}$  satisfies the  $(p, q)_r$  property if among any  $p$  elements of  $\mathcal{F}$ , at least  $r$  of the  $q$ -tuples intersect. They showed that  $\text{HD}_d(p, q)_r \leq p - q + 1$  holds for all  $r > \binom{p}{q} - \binom{p+1-d}{q+1-d}$ ; however, this is far from being tight.

In this paper we present improved asymptotic upper bounds on  $\text{HD}_d(p, q)_r$  which hold when only a tiny portion of the  $q$ -tuples intersect. In particular, we show that for  $p, q$  sufficiently large,  $\text{HD}_d(p, q)_r \leq p - q + 1$  holds with  $r = \frac{1}{p^{2d}} \binom{p}{q}$ . Our bound misses the known lower bound for the same piercing number by a factor of less than  $pq^d$ .

Our results use Kalai’s Upper Bound Theorem for convex sets, along with the Hadwiger-Debrunner theorem and the recent improved upper bound on  $\text{HD}_d(p, q)$  mentioned above.

## 1 Introduction

Throughout this paper,  $\mathcal{F}$  denotes a finite family of compact convex sets in  $\mathbb{R}^d$ ,  $p, q \in \mathbb{N}$  satisfy  $p \geq q \geq d + 1$ , and  $|\mathcal{F}| \geq p$ .  $\mathcal{F}$  is said to satisfy the  $(p, q)$  property if among any  $p$  elements of  $\mathcal{F}$  there is a  $q$ -tuple with a non-empty intersection. We say that  $\mathcal{F}$  is *pierced* by  $S \subset \mathbb{R}^d$  if any  $A \in \mathcal{F}$  satisfies  $A \cap S \neq \emptyset$ . The smallest cardinality of a set that pierces  $\mathcal{F}$  is called the *piercing number* of  $\mathcal{F}$ . We call  $\mathcal{F}$  *t-degenerate* if all elements of  $\mathcal{F}$  except at most  $t$  can be pierced by a single point. Otherwise,  $\mathcal{F}$  is called *non-t-degenerate*.

The classical Helly’s theorem asserts that if  $\mathcal{F}$  satisfies the  $(d + 1, d + 1)$  property (namely, if any  $d + 1$  elements of  $\mathcal{F}$  have a non-empty intersection), then the piercing number of  $\mathcal{F}$  is 1.

In 1957, Hadwiger and Debrunner [4] considered a natural generalization of Helly’s theorem to  $(p, q)$  properties. Let  $\text{HD}_d(p, q)$  be the maximum piercing number taken over all families  $\mathcal{F}$  of at least  $p$  compact convex sets in  $\mathbb{R}^d$  that satisfy the  $(p, q)$  property. Is  $\text{HD}_d(p, q)$  necessarily bounded for all  $p \geq q \geq d + 1$ ? (It is easy to see that  $\text{HD}_d(p, q)$  can be unbounded for  $q \leq d$ .)

Hadwiger and Debrunner showed that for all  $p \geq q \geq d + 1$  we have  $\text{HD}_d(p, q) \geq p - q + 1$ , and that equality is attained for any  $(p, q)$  such that  $q > \frac{d-1}{d}p + 1$  (and in particular, in  $\mathbb{R}^1$  equality is attained for all  $p \geq q \geq 2$ ). In a celebrated result from 1992, Alon and Kleitman [1] proved the Hadwiger-Debrunner conjecture, obtaining the upper bound  $\text{HD}_d(p, q) = \tilde{O}(p^{d^2+d})$ .

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However, as mentioned in [1], this bound is very far from being tight. The best currently known lower bound (implicitly implied by a result of Bukh et al. [3]), is  $\text{HD}_d(p, q) = \Omega\left(\frac{p}{q} \log^{d-1} \frac{p}{q}\right)$ .

Since the Alon-Kleitman theorem, several papers aimed at obtaining improved bounds on  $\text{HD}_d(p, q)$  for various values of  $(p, q, d)$ . The most notable result of this kind is by Kleitman et al. [7] who showed that  $\text{HD}_2(4, 3) \leq 13$  (compared to the upper bound of 345 obtained in [1]). Recently, it was shown in [6] that  $\text{HD}_d(p, q) \leq p - q + 2$  for all  $\varepsilon > 0$ ,  $p \geq p_0(\varepsilon)$ , and  $q > p^{\frac{d-1}{d} + \varepsilon}$ . The best currently known upper bound that holds for all  $q$ ,  $\text{HD}_d(p, q) = \tilde{O}(p^{d \cdot \frac{q-1}{q-d}})$  (also shown in [6]), is apparently far from being tight.

In an attempt to obtain improved bounds on  $\text{HD}_d(p, q)$  by refining the  $(p, q)$  property, Montejano and Soberón [9] introduced the following definition: A family  $\mathcal{F}$  is said to satisfy the  $(p, q)_r$  property if among any  $p$  elements of  $\mathcal{F}$ , at least  $r$  of the  $q$ -tuples intersect.  $\text{HD}_d(p, q)_r$  is defined as the maximal piercing number taken over all families that satisfy the  $(p, q)_r$  property. The main result of [9] is:

**Theorem 1.1** ([9]). *For any  $d$ ,*

$$\text{HD}_d(p, q)_r \leq p - q + 1 \tag{1}$$

*holds for all  $r > \binom{p}{q} - \binom{p+1-d}{q+1-d}$ .*

The proof of Theorem 1.1 uses a nice geometric argument. As mentioned in [9], the upper bound of Theorem 1.1 is far from being tight. Moreover, the value of  $r$  in the theorem is rather large – almost all the  $\binom{p}{q}$   $q$ -tuples are required to intersect.

In this paper we present improved upper bounds on  $\text{HD}_d(p, q)_r$ . For  $p, q$  sufficiently large (as function of  $d$ ), our bounds hold already when  $r$  is a tiny fraction of  $\binom{p}{q}$ . Our main result is the following:

**Theorem 1.2.**  $\text{HD}_d(p, q)_r$  *satisfy:*

$$1. \text{ For all } p \geq q \geq d+1 \text{ and } r \geq \Theta_d\left(\binom{\frac{d-1}{d}p}{q-d}\binom{\frac{p}{d}}{d}\right),$$

$$\text{HD}_d(p, q)_r \leq \min(p - q + 1, \frac{p}{d} - 1).$$

$$2. \text{ For any } \varepsilon > 0, \text{ any } p \geq q \geq d+1 \text{ such that } p > p_0(\varepsilon) \text{ and all } r \geq \Theta_{d,\varepsilon}\left(\frac{p^{\left(\frac{d-1}{d}+\varepsilon\right)q+1}}{(q-d)!}\right),$$

$$\text{HD}_d(p, q)_r \leq \min(p - q + 1, p - p^{\frac{d-1}{d} + \varepsilon} + 2).$$

*Here,  $\Theta_d(\cdot)$  hides a multiplicative factor that may depend on  $d$ .*

The latter bound on  $r$  is not far from being tight, as an explicit example presented in [9] (which we recall below) yields a lower bound of  $r = \Omega\left(\frac{p^{\left(\frac{d-1}{d}+\varepsilon\right)(q-1)+1}}{(q-1)!}\right)$  for the same piercing number. The upper and lower bounds differ by a multiplicative factor of  $\frac{p^{\frac{d-1}{d} + \varepsilon}(q-1)!}{(q-d)!}$ , which is smaller than  $pq^{d-1}$  for all  $\varepsilon \leq \frac{1}{d}$ .

We note that for  $p, q$  sufficiently large (as function of  $d$ ), the condition in (1) is equivalent to  $r \geq \frac{\binom{p}{q}}{c^q}$  for  $c > 1$  that depends only on  $d$ , and the condition in (2) (for  $\varepsilon = \frac{1}{2}$ ) is stronger than the condition  $r \geq \frac{\binom{p}{q}}{p^{\frac{q}{2d}}}$  stated in the abstract. This means that the assertion of Theorem 1.2 holds already when  $r$  is an exponentially (in  $q$ ) small fraction of  $\binom{p}{q}$ .

The proof of Theorem 1.2 uses Kalai's Upper Bound Theorem for convex sets [5], combined with the Hadwiger-Debrunner theorem and the recent improved upper bound on  $\text{HD}_d(p, q)$  obtained in [6].

In view of Theorem 1.2(1), it is natural to ask whether a smaller value of  $r$  is sufficient if we allow  $\text{HD}_d(p, q)_r$  to be larger than  $\frac{p}{d}$  (but still smaller than  $p - q$ ). We partially answer this question in the following generalization of Theorem 1.1.

**Theorem 1.3.** *For any  $p \geq q \geq d + 1$  and  $0 \leq k \leq p - q - 1$ , denote by  $m_0(k)$  the smallest integer  $m$  such that  $\binom{m+1}{2} \geq \frac{(p-q-k-1)(p-q+k+2)}{2} + 1$ . Let  $\mathcal{F}$  be a non- $(p - q)$ -degenerate family of compact convex sets in  $\mathbb{R}^d$  that satisfies the  $(p, q)_r$  property, with*

$$r \geq \binom{p}{q} - \binom{p-d+1}{q-d+1} + 1 + \binom{q-d-2+m_0(k)}{q-d} + \binom{q-d-1+m_0(k)}{q-d+1}. \quad (2)$$

*Then  $\mathcal{F}$  can be pierced by at most  $k + 2$  points.*

Note that in the case  $k = p - q - 1$ , Theorem 1.3 reduces to Theorem 1.1. The proof of Theorem 1.3 uses a bootstrapping technique based on the technique pioneered by Montejano and Soberón in [9]. The added value of Theorem 1.3 over Theorem 1.2 is demonstrated well for small values of  $(p, q)$ . For example, for  $(p, q, d) = (6, 3, 2)$ , Theorem 1.2 (actually, its proof) implies that  $r = 17$  is sufficient for assuring piercing by 2 points. Theorem 1.3 shows that actually  $r = 16$  suffice. In addition,  $r = 11$  is sufficient for piercing by 4 points.

We also show that the technique of Montejano and Soberón can be used to obtain an alternative proof of the Hadwiger-Debrunner theorem, which may be of independent interest due to its simplicity.

## 2 Proof of Theorem 1.2

As mentioned already, in the proof of Theorem 1.2 we use Kalai's Upper Bound Theorem for convex sets [5], the Hadwiger-Debrunner theorem [4], and the recent upper bound on  $\text{HD}_d(p, q)$  obtained in [6]. We state these results first.

**Theorem 2.1** ([5]). *Let  $\mathcal{F}$  be a family of  $p$  convex sets in  $\mathbb{R}^d$ . Denote by  $f_{q-1}$  the number of  $q$ -tuples of sets in  $\mathcal{F}$  whose intersection is non-empty. If  $f_{d+s} = 0$  for some  $s \geq 0$  then for any  $q > 0$ ,*

$$f_{q-1} \leq \sum_{i=0}^d \binom{s}{q-i} \binom{p-s}{i}.$$

**Theorem 2.2** ([4]). *For  $p \geq q \geq d + 1$  such that  $q > \frac{d-1}{d}p + 1$ ,*

$$\text{HD}_d(p, q) = p - q + 1.$$

**Theorem 2.3** ([6]). *Let  $\varepsilon > 0$ . There exists  $p_0(\varepsilon, d)$  such that for any  $p \geq q \geq d + 1$  with  $p \geq p_0$  and  $q \geq p^{\frac{d-1}{d}+\varepsilon}$ , we have*

$$\text{HD}_d(p, q) \leq p - q + 2.$$

The intuition behind the proof is simple. Theorems 2.2 and 2.3 yield a strong bound on the piercing number for a family that satisfies the  $(p, q)$  property with a 'large'  $q$ . In order to apply them, we need to 'enlarge'  $q$ , and this is done using Theorem 2.1. Specifically, if some family  $\mathcal{F}'$  of  $p$  convex sets contains 'many' intersecting  $q$ -tuples, Theorem 2.1 allows to deduce that it also contains an intersecting  $(q + k)$ -tuple, for an appropriate value of  $k$ . This implies that if a family  $\mathcal{F}$  satisfies the  $(p, q)_r$  property, then it must satisfy the  $(p, q + k)$  property, for  $k = k(r)$ . Applying this with a sufficiently large  $r$ , we replace  $q$  with a sufficiently large  $q + k$ , and then apply an improved bound on the piercing number that follows from Theorem 2.2 or Theorem 2.3.

## 2.1 Proof of Theorem 1.2(1)

We need the following lemma:

**Lemma 2.4.** *Let  $p \geq q \geq d + 1$ , and let  $1 \leq f(p) \leq \frac{p}{d} - 1$ . If*

$$r \geq r_0 := \sum_{i=0}^d \binom{p - f(p) - d}{q - i} \binom{f(p) + d}{i} + 1,$$

*then  $\text{HD}_d(p, q)_r \leq f(p)$ .*

*Proof.* Let  $\mathcal{F}$  be a family of compact convex sets in  $\mathbb{R}^d$  that satisfies the  $(p, q)_r$  property for some  $r \geq r_0$ . Put  $k = p - q - f(p) + 1$ . Note that  $\frac{d-1}{d}p - q + 2 \leq k \leq p - q$ . By the choice of  $r_0$ , Theorem 2.1 implies that  $\mathcal{F}$  satisfies the  $(p, q + k)$  property. As  $q + k \geq \frac{d-1}{d}p + 2$ , Theorem 2.2 implies that the piercing number of  $\mathcal{F}$  is at most  $p - (q + k) + 1 = f(p)$ , as asserted.  $\square$

*Proof of Theorem 1.2(1).* First, note that if  $p - q + 1 < \frac{p}{d} - 1$ , then  $q > \frac{d-1}{d}p + 2$ , and thus Theorem 2.2 implies  $\text{HD}_d(p, q)_r \leq p - q + 1$  even for  $r = 1$ . Hence, we may assume  $\frac{p}{d} - 1 \leq p - q + 1$ . Substituting  $f(p) = \frac{p}{d} - 1$  into Lemma 2.4, we get  $\text{HD}_d(p, q)_r \leq f(p) = \frac{p}{d} - 1$  for all

$$\begin{aligned} r &\geq \sum_{i=0}^d \binom{p - (\frac{p}{d} - 1) - d}{q - i} \binom{(\frac{p}{d} - 1) + d}{i} + 1 \\ &= \binom{\frac{d-1}{d}p + 1 - d}{q} + \binom{\frac{d-1}{d}p + 1 - d}{q-1} \cdot \left(\frac{p}{d} + d - 1\right) + \dots + \binom{\frac{d-1}{d}p + 1 - d}{q-d} \binom{\frac{p}{d} + d - 1}{d} \\ &= O_d \left( \binom{\frac{d-1}{d}p}{q-d} \binom{\frac{p}{d}}{d} \right), \end{aligned}$$

as asserted.  $\square$

**Remark 2.5.** *Note that Lemma 2.4 actually supplies a sequence of upper bounds on  $r$ , which correspond to any desired piercing number between 1 and  $\frac{p}{d} - 1$ . Piercing numbers larger than  $\frac{p}{d} - 1$  are treated in Section 3.*

## 2.2 Proof of Theorem 1.2(2)

The proof of Theorem 1.2(2) is similar to the proof of Theorem 1.2(1), with Theorem 2.3 replacing Theorem 2.2.

*Proof of Theorem 1.2(2).* Let  $\varepsilon > 0$ , and let  $p > p_0(\varepsilon)$  where  $p_0$  is chosen to satisfy the hypothesis of Theorem 2.3.

First, consider the case  $p - q + 1 < p - p^{\frac{d-1}{d} + \varepsilon} + 1$ . Recall that by assumption,  $\mathcal{F}$  satisfies the  $(p, q)_r$  property with

$$r \geq \Theta_{d,\varepsilon} \left( \frac{p^{(\frac{d-1}{d} + \varepsilon)q+1}}{(q-d)!} \right), \quad (3)$$

and thus, also with

$$r = \sum_{i=0}^d \binom{q-d}{q-i} \binom{p-q+d}{i} + 1$$

(actually, this is assured by taking the implicit factor in  $\Theta_{d,\varepsilon}(\cdot)$  to be sufficiently large). By Theorem 2.1, the latter implies that  $\mathcal{F}$  satisfies the  $(p, q + 1)$  property. As in this case,  $q > p^{\frac{d-1}{d} + \varepsilon}$ , Theorem 2.3 implies that  $\mathcal{F}$  can be pierced with at most  $p - (q + 1) + 2 = p - q + 1$  points, as asserted. Hence, we may assume  $p - p^{\frac{d-1}{d} + \varepsilon} + 1 \leq p - q + 1$ .

Let  $k = p^{\frac{d-1}{d}+\varepsilon} - q$ . Since by assumption,  $\mathcal{F}$  satisfies the  $(p, q)_r$  property with  $r$  that satisfies (3), in particular  $\mathcal{F}$  satisfies the  $(p, q)_r$  property with

$$\begin{aligned} r &= \sum_{i=0}^d \binom{p^{\frac{d-1}{d}+\varepsilon} - d - 1}{q - i} \binom{p - p^{\frac{d-1}{d}+\varepsilon} + d + 1}{i} + 1 \\ &= \sum_{i=0}^d \binom{q + k - d - 1}{q - i} \binom{p - q - k + d + 1}{i} + 1. \end{aligned}$$

By Theorem 2.1, this implies that  $\mathcal{F}$  satisfies the  $(p, q + k)$  property. As  $q + k = p^{\frac{d-1}{d}+\varepsilon}$ , Theorem 2.3 implies that  $\mathcal{F}$  can be pierced by at most  $p - p^{\frac{d-1}{d}+\varepsilon} + 2$  points. This completes the proof.  $\square$

**Remark 2.6.** As in Section 2.1, a similar argument (using Theorem 2.3 instead of Theorem 2.2) shows that for any  $p > p_0(\varepsilon)$  and any  $1 \leq f(p) \leq p - p^{\frac{d-1}{d}+\varepsilon} + 2$ , we have  $\text{HD}(p, q)_r \leq f(p)$  for all

$$r > \sum_{i=0}^d \binom{p - f(p) + 1 - d}{q - i} \binom{f(p) - 1 + d}{i}.$$

The upper bound on  $r$  asserted in Theorem 1.2(2) is not far from being optimal, as demonstrated by the following example (presented in [9]).

**Example.** Let  $\mathcal{F}$  be a family composed of  $p - p^{\frac{d-1}{d}+\varepsilon} + 3$  pairwise disjoint sets and  $p^{\frac{d-1}{d}+\varepsilon} - 3$  copies of a convex set that contains all of them. An easy computation shows that  $\mathcal{F}$  satisfies the  $(p, q)_r$  property for

$$r = \binom{p^{\frac{d-1}{d}+\varepsilon} - 3}{q} + (p - p^{\frac{d-1}{d}+\varepsilon} + 3) \cdot \binom{p^{\frac{d-1}{d}+\varepsilon} - 3}{q - 1} = \Theta\left(\frac{p^{(\frac{d-1}{d}+\varepsilon)(q-1)+1}}{(q-1)!}\right),$$

while it clearly cannot be pierced by  $p - p^{\frac{d-1}{d}+\varepsilon} + 2$  points.

A similar example, with  $p - f(p) + 2$  instead of  $p^{\frac{d-1}{d}+\varepsilon}$ , shows that the upper bound on  $r$  asserted in Remark 2.6 is also near tight.

Finally, we note that in dimension 1, the *exact relation* between the  $(p, q)_r$  property and the piercing number can be obtained easily using the Upper Bound Theorem.

**Proposition 2.7.** For  $p \geq q \geq 2$ , let  $\mathcal{F}$  be a family of segments on the real line that satisfies the  $(p, q)_r$  property. If

$$r \geq \binom{p - k - 2}{q} + (k + 2) \binom{p - k - 2}{q - 1} + 1, \quad (4)$$

then  $\mathcal{F}$  can be pierced by  $k + 1$  points. Conversely, there exists a family  $\mathcal{F}_0$  that satisfies the  $(p, q)_r$  property with  $r = \binom{p - k - 2}{q} + (k + 2) \binom{p - k - 2}{q - 1}$  and cannot be pierced by  $k + 1$  points.

*Proof.* By Theorem 2.1, if  $\mathcal{F}$  satisfies the  $(p, q)_r$  property with  $r$  that satisfies (4), then  $\mathcal{F}$  satisfies the  $(p, p - k)$  property. By Theorem 2.2 this implies that  $\mathcal{F}$  can be pierced by  $k + 1$  points.

For the other direction, let  $\mathcal{F}_0$  be a family that consists of  $k + 2$  distinct single-point sets, and  $p - k - 2$  copies of a segment that contains all the points. A straightforward computation shows that  $\mathcal{F}_0$  satisfies the  $(p, q)_r$  property with  $r = \binom{p - k - 2}{q} + (k + 2) \binom{p - k - 2}{q - 1}$ , but cannot be pierced by  $k + 1$  points.  $\square$

Proposition 2.7 will be useful for us in the next section.

### 3 Proof of Theorem 1.3

In the proof of Theorem 1.3 we use a bootstrapping based on the technique presented by Montejano and Soberón [9]. First we state a lemma of [9] on which we base our argument.

#### 3.1 The technique of [9] and an alternative proof of the Hadwiger-Debrunner theorem

**Lemma 3.1.** *For any family  $\mathcal{F}$  of convex sets in  $\mathbb{R}^d$ , there exist  $A_1, A_2, \dots, A_d \in \mathcal{F}$  and a line  $\ell$  such that if  $C \in \mathcal{F}$  intersects  $\cap_{i \neq j} A_i$  for all  $1 \leq j \leq d$  then  $C \cap \ell \neq \emptyset$ .*

Since our argument is partially based on the proof of Lemma 3.1 presented in [9], we recall the proof below. In the general case of families in  $\mathbb{R}^d$ , the proof of [9] uses topological techniques. As we do not use these parts of the proof of [9], we present the proof in the case of  $\mathbb{R}^2$  where the topological tools are not needed, and refer the reader to [2, Theorem 2.62] for sketch of the proof in the general case. For sake of clarity, we formulate explicitly the  $d = 2$  case of Lemma 3.1 whose proof we present.

**Lemma** (Lemma 3.1 for  $d = 2$ ). *For any family  $\mathcal{F}$  of convex sets in  $\mathbb{R}^2$ , there exist  $A, B \in \mathcal{F}$  and a line  $\ell$  such that if  $C \in \mathcal{F}$  satisfies  $A \cap C \neq \emptyset$  and  $B \cap C \neq \emptyset$  then  $C \cap \ell \neq \emptyset$ .*

*Proof.* If some  $A, B \in \mathcal{F}$  satisfy  $A \cap B = \emptyset$  then the assertion clearly holds with  $A, B$  and any line  $\ell$  that separates between  $A$  and  $B$ . Thus, we assume that  $A \cap B \neq \emptyset$  for all  $A, B \in \mathcal{F}$ .

For any pair  $A, B \in \mathcal{F}$  such that  $A \cap B \neq \emptyset$ , let  $\text{lexmax}(A, B)$  denote the lexicographic maximum of  $A \cap B$ . Let  $x_0 = \text{lexmin}\{\text{lexmax}(A, B) : A, B \in \mathcal{F}, A \cap B \neq \emptyset\}$  (i.e., the lexicographic minimum amongst  $\text{lexmax}(A, B)$ ), and let  $A, B \in \mathcal{F}$  be such that  $x_0 = \text{lexmax}(A, B)$ . Denote  $H = \{x \in \mathbb{R}^2 : x \geq_{\text{lex}} x_0\}$ , and let  $A' = A \cap H$ ,  $B' = B \cap H$ . As  $A', B'$  are convex sets and  $A' \cap B' = \{x_0\}$ , there exists a line  $\ell$  with  $x_0 \in \ell$  that separates between  $A' \setminus \{x_0\}$  and  $B' \setminus \{x_0\}$ . We claim that the assertion holds with  $A, B, \ell$ . To see this, we consider two cases:

1.  $C \cap (A \cap B) \neq \emptyset$ . We claim that  $x_0 \in C$ , and thus  $C \cap \ell \neq \emptyset$ . Assume to the contrary  $x_0 \notin C$ . Note that for any family of convex sets  $C_1, C_2, \dots, C_m \subset \mathbb{R}^2$  such that  $\cap_{i=1}^m C_i \neq \emptyset$ , there exist  $1 \leq k < l \leq m$  such that  $\text{lexmax}(\cap_{i=1}^m C_i) = \text{lexmax}(C_k \cap C_l)$ . (This is a straightforward application of Helly's theorem; see [8], Lemma 8.1.2). In the case  $\{C_1, \dots, C_m\} = \{A, B, C\}$ , by assumption  $x_1 := \text{lexmax}(A \cap B \cap C) \neq \text{lexmax}(A \cap B)$ , and thus w.l.o.g.  $x_1 = \text{lexmax}(A \cap C)$ . It follows that  $x_1 \in A \cap B$ , and thus,  $x_1 <_{\text{lex}} x_0 = \text{lexmax}(A \cap B)$ . A contradiction to the definition of  $x_0$ . Hence,  $x_0 \in C$ , as asserted.
2.  $C \cap (A \cap B) = \emptyset$ . As  $\text{lexmax}(A \cap C) >_{\text{lex}} x_0$ , we have  $(A' \setminus \{x_0\}) \cap C \neq \emptyset$ . Similarly, we have  $(B' \setminus \{x_0\}) \cap C \neq \emptyset$ . As  $\ell$  separates between  $A' \setminus \{x_0\}$  and  $B' \setminus \{x_0\}$ , this implies  $C \cap \ell \neq \emptyset$ , as asserted.

□

**Remark 3.2.** *When the proof of Case (1) of Lemma 3.1 is applied for a general  $d$  (as was done in [9] and as we do below), we define  $x_0 = \text{lexmin}\{\text{lexmax}(\cap_{i=1}^d A_i) : A_1, \dots, A_d \in \mathcal{F}, \cap_{i=1}^d A_i \neq \emptyset\}$  (where the system of coordinates is chosen such that all lexicographic maxima/minima are defined uniquely). We also replace  $A \cap B$  by  $\cap_{i=1}^d A_i$ , and replace ‘each of  $A$  and  $B$ ’ by ‘each of  $\cap_{i \neq j} A_i$ ,  $1 \leq j \leq d$ ’.*

The argument used in Case (1) above can be used to obtain a simple proof of the Hadwiger-Debrunner theorem (Theorem 2.2 above), as follows:

*Alternative proof of Theorem 2.2.* Let  $\mathcal{F}$  be a family of at least  $p$  compact convex sets in  $\mathbb{R}^d$  that satisfies the  $(p, q)$  property, and let  $x_0, A_1, A_2, \dots, A_d$  be chosen as in the proof of Lemma 3.1 and in Remark 3.2.

Consider the family  $\mathcal{G} = \{C \in \mathcal{F} : x_0 \notin C\}$ . We consider two cases:

- $|\mathcal{G}| \geq p - d$ . We claim that in this case,  $\mathcal{G}$  satisfies the  $(p - d, q - d + 1)$  property. Indeed, let  $C_1, C_2, \dots, C_{p-d} \in \mathcal{G}$ , and consider the family  $\{C_1, \dots, C_{p-d}, A_1, A_2, \dots, A_d\}$ . By assumption, it contains an intersecting  $q$ -tuple. This  $q$ -tuple cannot contain all of  $A_1, A_2, \dots, A_d$ , as by the argument of Case (1) above, each of  $C_1, \dots, C_{p-d}$  is disjoint with  $\cap_{i=1}^d A_i$ . Thus,  $\{C_1, C_2, \dots, C_{p-d}\}$  contains an intersecting  $(q - d + 1)$ -tuple.
- $|\mathcal{G}| = p - d - t$  for  $t > 0$ . By the same reasoning as in the previous case,  $\mathcal{G}$  contains an intersecting  $(q - d - t + 1)$ -tuple that can be pierced by a single point, and thus, it can be trivially pierced by  $(p - d - t) - (q - d - t + 1) + 1 = p - q$  points.

As  $\mathcal{F} \setminus \mathcal{G}$  is pierced by  $x_0$ , combining the two cases we get  $\text{HD}(p, q) \leq \max(\text{HD}(p - 2, q - 1) + 1, p - q + 1)$ . Since  $\text{HD}(d + 1, d + 1) = 1$  by Helly's theorem, it follows by induction that if  $q > \frac{d-1}{d}p + 1$  then  $\text{HD}(p, q) \leq p - q + 1$ .  $\square$

### 3.2 The bootstrapping technique

In [9], the authors show that if  $\mathcal{F}$  satisfies the  $(p, q)_r$  property with  $r > \binom{p}{q} - \binom{p+1-d}{q+1-d}$  then (in the notations of Lemma 3.1) the family  $\mathcal{G}' = \{C \cap \ell : C \in \mathcal{F}, x_0 \notin C\}$  satisfies the  $(p - d, q - d + 1)$  property, and thus, by Theorem 2.2 in dimension 1,  $\mathcal{F}$  can be pierced by  $p - q + 1$  points. In our bootstrapping argument, we show instead that the family  $\mathcal{G}'$  satisfies the  $(p - q + 1, 2)_{r'}$  property for a sufficiently large  $r'$ , and then an improved piercing number for  $\mathcal{F}$  can be derived from Proposition 2.7. We will use the following.

**Definition 3.3.** Let  $\mathcal{F}$  be a family of compact convex sets in  $\mathbb{R}^d$ ,  $|\mathcal{F}| \geq p$ , and let  $\ell$  be a line.  $\mathcal{F}$  is said to satisfy the  $(p, q)_r$  property through  $\ell$  if any  $p$ -tuple of sets in  $\mathcal{F}$  contains at least  $r$   $q$ -tuples that intersect on  $\ell$ .

**Lemma 3.4.** If a family  $\mathcal{F}$  satisfies the  $(p, 2)_{r_0}$  property through  $\ell$  where  $r_0 = \binom{p-k-2}{2} + (k + 2)\binom{p-k-2}{1} + 1$ , then  $\mathcal{F}$  can be pierced by  $k + 1$  points.

*Proof.* Let  $\mathcal{H} = \{C \in \mathcal{F} : C \cap \ell = \emptyset\}$ , and denote  $h = |\mathcal{H}|$ . The family  $\mathcal{F}' = \{C \cap \ell : C \in \mathcal{F} \setminus \mathcal{H}\}$  clearly satisfies the  $(p - h, 2)_{r_0}$  property. As

$$\begin{aligned} \binom{p-k-2}{2} + (k+2)\binom{p-k-2}{1} + 1 &\geq \binom{(p-h)-(k-h)-2}{2} \\ &\quad + ((k-h)+2)\binom{(p-h)-(k-h)-2}{1} + 1, \end{aligned}$$

it follows that  $\mathcal{F}'$  is a family of segments on  $\ell$  that satisfies the  $(p - h, 2)_{r'}$  property with  $r' = \binom{(p-h)-(k-h)-2}{2} + ((k-h)+2)\binom{(p-h)-(k-h)-2}{1} + 1$ . Thus, by Proposition 2.7,  $\mathcal{F}'$  can be pierced by  $k - h + 1$  points, and thus,  $\mathcal{F}$  can be pierced by  $k + 1$  points, as asserted.  $\square$

Now we are ready to prove Theorem 1.3.

*Proof of Theorem 1.3.* Let  $\mathcal{F}$  be a family that satisfies the assumption of the theorem, and let  $x_0, A_1, A_2, \dots, A_d, \ell$  be chosen as in the proof of Lemma 3.1, i.e.,  $\{A_i\}_{i=1}^d$  is the  $d$ -tuple in which the  $\text{lexmin}(\text{lexmax}(\cdot))$  is attained. Denote  $\mathcal{F}' = \{C \in \mathcal{F} : x_0 \notin C\}$ . We want to show that  $\mathcal{F}'$  satisfies the  $(p - q + 1, 2)_{r_0}$  property through  $\ell$ , where  $r_0 = \binom{(p-q+1)-k-2}{2} + (k+2)\binom{(p-q+1)-k-2}{1} + 1$ . By Lemma 3.4, this would imply that  $\mathcal{F}'$  can be pierced by  $k + 1$  points, and thus,  $\mathcal{F}$  can be pierced by  $k + 2$  points, as asserted.

By the choice of  $m_0(k)$ , it is sufficient to show that  $\mathcal{F}'$  satisfies the  $(p - q + 1, 2)_{r'}$  property through  $\ell$ , where  $r' = \binom{m_0(k)+1}{2}$ . Furthermore, by Lemma 3.1 it is sufficient to show that among any  $p - q + 1$  elements of  $\mathcal{F}$  there exist at least  $\binom{m_0(k)+1}{2}$  distinct pairs of elements that intersect  $\cap_{i \neq j} A_i$  for all  $1 \leq j \leq d$ .

As  $\mathcal{F}$  is non- $(p-q)$ -degenerate, we have  $|\mathcal{F}'| \geq p-q+1$ . Let  $\mathcal{C} = \{C_1, C_2, \dots, C_{p-q+1}\} \subset \mathcal{F}'$ , and let  $\mathcal{D} = \{D_1, D_2, \dots, D_{q-d-1}\} \subset \mathcal{F}$  such that  $\mathcal{D} \cap (\mathcal{C} \cup \{A_1, A_2, \dots, A_d\}) = \emptyset$ . We have  $|\mathcal{C} \cup \mathcal{D} \cup \{A_1, A_2, \dots, A_d\}| = p$ , and thus, the family  $\mathcal{C} \cup \mathcal{D} \cup \{A_1, A_2, \dots, A_d\}$  contains at least  $r$  intersecting  $q$ -tuples.

Note that  $q$ -tuples of elements of  $\mathcal{C} \cup \mathcal{D} \cup \{A_1, A_2, \dots, A_d\}$  can be divided into three groups:

1.  $q$ -tuples that contain less than  $d-1$  of the sets  $A_1, \dots, A_d$ .
2.  $q$ -tuples that contain exactly  $d-1$  of the sets  $A_1, \dots, A_d$ .
3.  $q$ -tuples that contain all the sets  $A_1, \dots, A_d$ .

We observe that none of the intersecting  $q$ -tuples belong to the third group, as by the proof of Lemma 3.1 above (specifically, by Lemma 8.1.2 in [8] that applies for a general  $d$ ), all elements of  $\mathcal{C}$  are disjoint with  $\cap_{i=1}^d A_i$ , and  $\mathcal{D}$  contains only  $q-d-1$  elements. This implies that the total number of intersecting  $q$ -tuples is at most  $\binom{p}{q} - \binom{p-d}{q-d}$ . Furthermore, since  $r$  satisfies (2), the number of *non-intersecting*  $q$ -tuples in groups 1 and 2 is at most

$$\begin{aligned} & \binom{p}{q} - \binom{p-d}{q-d} - \left( \binom{p}{q} - \binom{p-d+1}{q-d+1} + 1 + \binom{q-d-2+m_0(k)}{q-d} + \binom{q-d-1+m_0(k)}{q-d+1} \right) \\ &= \binom{p-d}{q-d+1} - \left( \binom{q-d-2+m_0(k)}{q-d} + \binom{q-d-1+m_0(k)}{q-d+1} + 1 \right). \end{aligned} \quad (5)$$

For each  $\{S_1, S_2, \dots, S_{q-d+1}\} \subset \mathcal{C} \cup \mathcal{D}$ , we define a  $d$ -tuple

$$P_{\{S_1, \dots, S_{q-d+1}\}} = \{ \{S_1, \dots, S_{q-d+1}, A_2, A_3, \dots, A_d\}, \{S_1, \dots, S_{q-d+1}, A_1, A_3, \dots, A_d\}, \dots, \{S_1, \dots, S_{q-d+1}, A_1, A_2, \dots, A_{d-1}\} \}.$$

Denote by  $P'$  the set of all  $\{S_1, S_2, \dots, S_{q-d+1}\} \subset \mathcal{C} \cup \mathcal{D}$  for which all  $d$  elements of  $P_{\{S_1, \dots, S_{q-d+1}\}}$  are intersecting. We claim that

$$|P'| \geq \binom{q-d-2+m_0(k)}{q-d} + \binom{q-d-1+m_0(k)}{q-d+1} + 1. \quad (6)$$

Indeed, note that the  $q$ -tuples in group 2 are naturally divided into  $d$  classes according to the set  $A_i$  they miss. Each class consists of  $\binom{p-d}{q-d+1}$   $q$ -tuples. It is clear that for a given number of intersecting  $q$ -tuples,  $|P'|$  is minimized when all non-intersecting  $q$ -tuples of group 2 belong to the same class. In that case,  $|P'|$  equals to the number of remaining elements in that class, and thus by Equation (5),

$$|P'| \geq \binom{p-d}{q-d+1} - \left( \binom{p-d}{q-d+1} - \left( \binom{q-d-2+m_0(k)}{q-d} + \binom{q-d-1+m_0(k)}{q-d+1} + 1 \right) \right),$$

meaning that (6) holds.

By the definition of  $P'$ , each element in  $P'$  contains  $q-d+1$  sets that intersect  $\cap_{i \neq j} A_i$ , for all  $1 \leq j \leq d$ . As  $|\mathcal{D}| = q-d-1$ , at least two of these sets belong to  $\mathcal{C}$ . Hence, each element of  $P'$  contains at least one pair of elements in  $\mathcal{C}$  that intersect  $\cap_{i \neq j} A_i$ , for all  $1 \leq j \leq d$ . Recall that we want to prove that there are at least  $\binom{m_0(k)+1}{2}$  such pairs.

It is easy to see that for a given number of elements in  $P'$ , the number of distinct pairs  $(C, C') \in \mathcal{C}$  contained in elements of  $P'$  is minimized when these elements are ‘packed together’. In particular, the maximal possible number of elements in  $P'$  such that the number of distinct



pairs is *smaller than*  $\binom{m_0(k)+1}{2}$  is attained when we take some  $C_1, C_2, \dots, C_{m_0(k)+1} \in \mathcal{C}$ , and define

$$P'' = \{S \subset \mathcal{C} \cup \mathcal{D} : (|S| = q - d + 1) \wedge (S \cap \mathcal{C} \subset \{C_1, C_2, \dots, C_{m_0(k)+1}\}) \wedge (\{C_{m_0(k)}, C_{m_0(k)+1}\} \not\subset S)\}. \quad (7)$$

In this case, we have  $|P''| = \binom{q-d-2+m_0(k)}{q-d} + \binom{q-d-1+m_0(k)}{q-d+1}$ . Indeed, since  $|\mathcal{D}| = q - d - 1$ , then among the  $(q - d + 1)$ -tuples  $S$  for which  $S \cap \mathcal{C} \subset \{C_1, C_2, \dots, C_{m_0(k)+1}\}$ , there are  $\binom{q-d-1+m_0(k)}{q-d+1}$  that do not include  $C_{m_0(k)+1}$ , and  $\binom{q-d-2+m_0(k)}{q-d}$  that include  $C_{m_0(k)+1}$  and miss  $C_{m_0(k)}$ . Therefore, Equation (6) implies that  $\mathcal{C}$  must contain at least  $\binom{m_0(k)+1}{2}$  distinct pairs that intersect  $\cap_{i \neq j} A_i$  for all  $1 \leq j \leq d$ , and thus, by Lemma 3.1,  $\mathcal{F}'$  satisfies the  $(p - q + 1, 2)_{\binom{m_0(k)+1}{2}}$  property through  $\ell$ . This completes the proof.  $\square$

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## References

- [1] N. Alon and D.J. Kleitman, Piercing convex sets and the Hadwiger-Debrunner  $(p, q)$ -problem, *Adv. Math.*, **96**(1) (1992), pp. 103-112.
- [2] N. Amenta, J. A. De Loera, and P. Soberón, Helly's Theorem: New variations and applications, preprint, 2015. Available at: <http://arxiv.org/abs/1508.07606>.
- [3] B. Bukh, J. Matoušek, and G. Nivasch, Lower bounds for weak epsilon-nets and stair-convexity, *Israel J. Math.*, **182**(1) (2011), pp. 199-228.
- [4] H. Hadwiger and H. Debrunner. Über eine variante zum hellyschen satz, *Arch. Math.*, **8**(4) (1957), pp. 309-313.
- [5] G. Kalai, Intersection patterns of convex sets, *Israel J. Math.*, **48** (1984), pp. 161-174.
- [6] C. Keller, S. Smorodinsky, and G. Tardos, On Max-Clique for intersection graphs of sets and the Hadwiger-Debrunner numbers, Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA) 2017, pp. 2254-2263. Available at: <http://arxiv.org/abs/1512.04026>.
- [7] D.J. Kleitman, A. Gyárfás, and G. Tóth, Convex sets in the plane with three of every four meeting, *Combinatorica*, **21**(2) (2001), pp. 221-232.
- [8] J. Matoušek, Lectures on discrete geometry, Springer-Verlag New York, Inc., Secaucus, NJ, USA, 2002.
- [9] L. Montejano and P. Soberón, Piercing numbers for balanced and unbalanced families, *Disc. Comput. Geom.* **45**(2) (2011), pp. 358-364.